Form factors for quasi-particles in $c = 1$ conformal field theory

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R A J van Elburg and K Schoutens

Van der Waals–Zeeman Institute and Institute for Theoretical Physics, University of Amsterdam,
Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

Abstract. The non-Fermi liquid physics at the edge of fractional quantum Hall systems is described by specific chiral conformal field theories with central charge $c = 1$. The charged quasi-particles in these theories have fractional charge and obey a form of fractional statistics. In this paper we study form factors, which are matrix elements of physical (conformal) operators, evaluated in a quasi-particle basis that is organized according to the rules of fractional exclusion statistics. Using the systematics of Jack polynomials, we derive selection rules for a special class of form factors. We argue that finite-temperature Green functions can be evaluated via systematic form factor expansions, using form factors such as those computed in this paper and thermodynamic distribution functions for fractional exclusion statistics. We present a specific case study where we demonstrate that the form factor expansion shows a rapid convergence.

1. Introduction and summary

Shortly after the first observation of the fractional quantum Hall (fqH) effect [1], Laughlin proposed that this phenomenon has its origin in a new state of matter, which is formed by electrons but nevertheless admits excitations of fractional charge [2]. The experimental evidence for the existence of fractionally charged excitations includes the results of shot-noise measurements for charge transport through fqH samples [3].

The unusual properties of the fqH quasi-particles do not stop at their fractional charge; in addition, the (bulk and edge) quasi-particles exhibit various forms of fractional quantum statistics [4–7].

In a previous paper [8] we initiated a programme that aims at computing various transport properties of fqH systems in a formalism that makes direct reference to the (fractionally) charged quasi-particles. In particular, we presented a quasi-particle basis of the edge conformal field theory (CFT) for the $\nu = \frac{1}{p}$ principal Laughlin states. We demonstrated that the CFT quasi-particles satisfy a form of fractional statistics that is closely related to Haldane's notion of 'fractional exclusion statistics' [6].

The CFT for the $\nu = 1/p$ fqH edges can be identified with a continuum ($N \to \infty$) limit of the Calogero–Sutherland (CS) model for quantum mechanics with inverse square exchange. The natural quasi-particles for the fqH system correspond to eigenstates of the CS Hamiltonian. In the work of many groups, the eigenstates of Hamiltonians of CS type have been understood in terms of Jack symmetric polynomials. In particular, Iso [9] presented two alternative Jack polynomial bases for the continuum CS theory. In section 3.4 below, we discuss the precise correspondence between these CS bases and the fqH basis we presented in [8].

It has been recognized by many groups that quantum many-body systems with inverse square exchange come close to being 'ideal gases of fractional statistics particles'. Supporting

this claim are the observations that (a) equilibrium thermodynamic quantities can be evaluated with the help of one-particle distribution functions for fractional statistics and (b) the zero-temperature correlation functions of simple operators (such as electron and hole operators) are mediated by intermediate states with a minimal number of propagating quasi-particles (see section 4).

When turning to finite temperature, one quickly discovers that the second 'free gas' property (b) no longer holds. There are important many-body effects which cannot be ignored when computing correlation functions of physical operators at finite temperature. This implies that great care needs to be taken when setting up arguments that link the T dependence of physical observables (I–V and shot-noise characteristics, in particular) to the fractional statistics of the fundamental charge carriers [10].

In this paper we turn to the problem of extending the formalism based on quasi-particles with fractional statistics to finite temperatures. We argue that specific finite-T correlation functions can be written in a so-called form factor expansion. Such expansions start with a term that refers to a minimal number of quasi-particles; to this leading term successive corrections are added that refer to more and more quasi-particles participating in a physical process described by the correlation function. To test the validity of the proposed expansion, we explicitly evaluate the form factor expansion for a specific finite-T Green function, collecting all terms that refer to one or two quasi-particles. The numerical results show a rather rapid convergence to the (known) exact expression.

The form factor expansion that we propose is similar in spirit to expressions that were proposed in [11] (see also [12]) in the context of integrable quantum field theories whose structure is set by a factorizable scattering matrix. Despite this similarity, the two approaches are rather different: in our approach we do not rely on scattering data or on the associated form factor axioms, but instead perform explicit computations in a CFT that is regularized by the finite size L of the spatial direction.

This paper is organized as follows. In section 2 we review results obtained in our earlier paper [8]. We present a basis for the edge theories for the $\nu = \frac{1}{p}$ fqH state, employing edge electrons and edge quasi-holes as the fundamental charged quasi-particles. We describe the associated one-particle thermodynamic distribution functions and discuss the fundamental duality between the two types of quasi-particles.

In section 3 we discuss the continuum CS models and the associated Jack polynomial bases as first given by Iso. We give an explicit one-to-one connection between the states in the CS basis and the states in the fqH basis.

In section 4 we turn to form factors. In section 4.1 we discuss those relevant for $T = 0$, while in section 4.2 we present simple examples of form factors that contribute at non-zero T. The expressions that we obtain make clear that the simple picture of 'an ideal gas' breaks down at finite temperature. A rather general set of selection rules is presented in section 4.3, while in section 4.4 we briefly discuss the extent to which our explicit results can be understood from an axiomatic approach based on a factorized S-matrix.

In section 5 we compute one- and two-particle diagonal form factors for the 'edge electron counting operator'. These form factors are then used to evaluate the leading terms in a form factor expansion for the finite-T Green function that determines the I–V characteristics of edge tunnelling processes.

In section 6 we present our conclusions. In the appendices we specify algebraic properties of the charged edge operators for $\nu = \frac{1}{2}$, we summarize some relevant results on Jack polynomials and we provide a proof of the form factor selection rules presented in section 4.3.

2. Quasi-particle basis of fqH edge theories

2.1. fqH edge theories as CFT

It is well known that many (though not all!) aspects of the low-energy dynamics of fqH systems can be captured by an effective edge theory. These edge theories are so-called chiral Luttinger liquids or chiral CFTs. For the specific case of a principal Laughlin state at filling $\nu = 1/p$, the CFT describing a single edge (in isolation) is a specific $c = 1$ CFT.

The standard interpretation of this effective theory is in terms of a chiral boson, which is identified with a quantized density wave (magnetoplasmon) along the edge of the fqH sample. By exploiting the relation between the chiral anomaly and the quantized Hall conductance, one finds that the chiral boson field is compactified on a circle of radius $R^2 = p$ [13]. This construction leads directly to a space of states (the so-called chiral Hilbert space) with partition function

$$Z^{1/p}(q) = \sum_{Q=-\infty}^{\infty} \frac{q^{Q^2/2p}}{(q)_{\infty}} \tag{2.1}$$

with $(q)_{\infty} = \prod_{l=1}^{\infty}(1 - q^l)$ and $q = \exp\left[-\beta \frac{2\pi}{L} \frac{1}{\rho_0}\right]$. (The one-particle energies are spaced by $l\frac{2\pi}{L}\frac{1}{\rho_0}$ with l an integer and ρ_0 the density of states per unit length, $\rho_0 = (\hbar v_{\text{F}})^{-1}$.)

In equation (2.1), the parameter Q is the electric $U(1)$ charge in units of $\frac{e}{p}$. The Hilbert space is obtained as a collection of charge sectors Q. Within each sector, there is a leading state of minimal energy $\frac{Q^2}{2p}$; all other states in that sector are reached via the application of (neutral) Fourier modes of the density operator. Together, these modes form a $U(1)$ Kac–Moody algebra, and the factor $\frac{1}{(q)_{\infty}}$ is the well known character of a highest-weight module of this affine algebra.

In our earlier paper [8], we proposed that the CFT for the $\nu = \frac{1}{p}$ fqH edge can be interpreted in terms of a set of fundamental charged quasi-particles. We have worked out a formulation in terms of edge electrons (of charge $-e$) and edge quasi-holes (of charge $+\frac{e}{p}$). Our main motivation has been that, using this novel formulation, one learns how to understand some of the unusual and spectacular phenomenology of the fqH systems as manifestations of unusual properties (fractional charge and fractional statistics, in particular) of their fundamental quasi-particles.

2.2. The fqH quasi-particle basis

In [8], we demonstrated how the collection of states (2.1) can be understood as a collection of multi-particle states, the fundamental (quasi-)particles being the edge electron and the edge quasi-hole, of charge $Q = -p$ and $Q = +1$, respectively.

The edge electron and quasi-holes are described by the conformal, primary fields

$$J^{(-p)}(z) = \sum_t J_{-t}^{(-p)} z^{t-p/2} \qquad \phi^+(z) = \sum_s \phi_{-s}^+ z^{s-1/2p}. \tag{2.2}$$

Clearly, one can employ the Fourier modes $J_{-t}^{(-p)}$ and ϕ_{-s}^+ as 'creation operators' for the corresponding quasi-particles. (The reason why we use quotation marks here will soon become clear.) In [8] we identified a collection of multi-J, multi-ϕ states which together form a basis for the chiral Hilbert space. It is given by the states (from now on we omit the charge superindex on the operators J, ϕ)

$$J_{-(2M-1)p/2+Q-m_M} \cdots J_{-p/2+Q-m_1} \phi_{-(2N-1)1/2p-Q/p-n_N} \cdots \phi_{-1/2p-Q/p-n_1} | Q \rangle \tag{2.3}$$

with

$$m_M \geqslant m_{M-1} \geqslant \cdots \geqslant m_1 \geqslant 0 \qquad n_N \geqslant n_{N-1} \geqslant \cdots \geqslant n_1 \geqslant 0$$

$$n_1 > 0 \qquad \text{if} \quad Q < 0$$

where $|Q\rangle$ denotes the lowest-energy state of charge $Q \frac{e}{p}$, with Q taking the values $-(p-1)$, $-(p-2), \ldots, -1, 0$.

The associated character identity is

$$Z^{1/p}(q) = \sum_{Q=-(p-1)}^{0} q^{Q^2/2p} Z_Q^{\text{qh}}(q) Z_Q^{\text{e}}(q) \tag{2.4}$$

where the factor $q^{Q^2/2p}$ takes into account the energy of the initial states and we denoted by $Z_Q^{\text{qh}}, Z_Q^{\text{e}}$ the partition functions for quasi-holes and electrons in the sector with vacuum charge Q. They are naturally written as

$$Z_Q^{\text{qh}} = \sum_{N=0}^{\infty} \frac{q^{\frac{1}{2p}(N^2+2QN)+(1-\delta_{Q,0})N}}{(q)_N} \qquad Z_Q^{\text{e}} = \sum_{M=0}^{\infty} \frac{q^{\frac{p}{2}M^2-QM}}{(q)_M} \tag{2.5}$$

with $(q)_L = \prod_{l=1}^{L}(1-q^l)$. The identity (2.4) can be rigorously established by employing partition counting theorems that are available in the mathematical literature (see [14] for a discussion).

An important special case is $p = 1$, where the quasi-particle basis (2.3) is the standard multi-particle basis in a theory of free, charged but spinless fermions.

2.3. Fractional statistics and duality

2.3.1. Fractional exclusion statistics. In a 1991 paper [6], Haldane proposed the notion 'fractional exclusion statistics', as a tool for the analysis of strongly correlated many-body systems. The central assumption that is made concerns the way a many-body spectrum is built by filling available one-particle states. In other words, it is assumed that the act of filling a one-particle state effectively reduces the dimension of the space of the remaining one-particle states by an amount g. The choices $g = 1, 0$ correspond to fermions and bosons, respectively. The thermodynamics for general 'g-ons', and, in particular, the appropriate generalization of the Fermi–Dirac distribution function, have been obtained in [15]. The so-called Isakov–Ouvry–Wu (IOW) equations

$$\bar{n}_g(\epsilon) = \frac{1}{[w(\epsilon) + g]} \qquad \text{with} \quad [w(\epsilon)]^g [1 + w(\epsilon)]^{1-g} = e^{\beta(\epsilon-\mu)} \tag{2.6}$$

provide an implicit expression for the one-particle distribution function $\bar{n}_g(\epsilon)$ for g-ons at temperature T and chemical potential μ. The solutions $\bar{n}_g(\epsilon)$ have the limiting value $\bar{n}_g^{\max} = \frac{1}{g}$ for $\epsilon \to -\infty$.

2.3.2. Spectral shift statistics and the fqH effect. In [8] we analysed the exclusion statistics behind the states (2.3) that form a basis for the chiral Hilbert space for a $\nu = \frac{1}{p}$ fqH edge. This analysis employed a technique, first proposed in [16], based on recursion relations satisfied by truncated chiral partition functions. The remarkable conclusions are that

- the 'microscopic' state-filling rules differ from those formulated by Haldane, but

- the state counting, and thereby the one-particle thermodynamic distribution functions, agree with those associated with fractional exclusion statistics, i.e. the distribution functions are solutions of the IOW equations.

The precise statement is that the edge electrons are described by the IOW distribution with $g = p$, while the edge quasi-hole states are thermally occupied according to the distribution with $\tilde{g} = 1/p$. It is important to remark that there is no mutual statistics between the two types of excitations.

For later reference, we list the explicit expressions for the distribution functions for the case $p = 2$,

$$\bar{n}_2(\epsilon) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1 + 4\,\mathrm{e}^{-\beta(\epsilon-\mu)}}}\right) \qquad \bar{n}_{1/2}(\epsilon) = \frac{2}{\sqrt{1 + 4\mathrm{e}^{2\beta(\epsilon-\mu)}}}. \qquad (2.7)$$

2.3.3. Duality. The distribution functions for fractional exclusion statistics with parameters g and $\tilde{g} = 1/g$ satisfy the following duality relation [17]:

$$g\,\bar{n}_g(\epsilon) = 1 - \tilde{g}\,\bar{n}_{\tilde{g}}(-\tilde{g}\epsilon). \qquad (2.8)$$

The interpretation of this result is that the $\tilde{g}$ quanta with positive energy act as holes in the ground state distribution of negative energy g-quanta.

Translating back to the $\nu = \frac{1}{p}$ fqH edge, we observe a fundamental duality between edge electrons and edge quasi-holes, in agreement with the physical interpretation of strong and weak backscattering limits of edge-to-edge tunnelling processes [18]. Under the duality, the removal of a single edge electron corresponds to the creation of a total of p quasi-holes.

This duality further implies that, when setting up a quasi-particle description for fqH edges, we can opt for (a) either quasi-holes or edge electrons, with energies over the full range $-\infty < \epsilon < \infty$ ('particle picture'), or (b) a combination of both types of quasi-particles, each having positive energies only ('excitation picture'). Option (b) is that realized in the fqH basis of section 2.2. In section 3 we shall discuss the CS bases proposed by Iso, which in a sense uses option (a).

2.4. Equilibrium quantities

2.4.1. Specific heat. The specific heat of a conformal field theory is well known to be proportional to the central charge c_{CFT}

$$\frac{C(T)}{L} = \gamma \rho_0 k_{\mathrm{B}}^2 T \qquad \gamma = \frac{\pi}{6}\, c_{\mathrm{CFT}} \qquad (2.9)$$

where $\rho_0 = (\hbar v_{\mathrm{F}})^{-1}$ is the density of states per unit length. In [8] we demonstrated that the fqH quasi-particle basis specified in (2.3) leads to (with $g = p$, $\tilde{g} = \frac{1}{p}$)

$$\gamma = \gamma_{g,+} + \gamma_{\tilde{g},+} \qquad (2.10)$$

with

$$\gamma_{g,+} = \partial_\beta \int_0^\infty \mathrm{d}\epsilon\,\epsilon\,\bar{n}_g(\epsilon) \qquad \gamma_{\tilde{g},+} = \partial_\beta \int_0^\infty \mathrm{d}\epsilon\,\epsilon\,\bar{n}_{\tilde{g}}(\epsilon). \qquad (2.11)$$

One finds that, while the individual contributions $\gamma_{g,+}$, $\gamma_{\tilde{g},+}$ depend on g, their sum is equal to $\frac{\pi}{6}$ for all g, in agreement with $c_{\mathrm{CFT}} = 1$. For $g = 2$, $\tilde{g} = \frac{1}{2}$ one has

$$\gamma_{2,+} = \frac{\pi}{6}\frac{2}{5} \qquad \gamma_{\frac{1}{2},+} = \frac{\pi}{6}\frac{3}{5}. \qquad (2.12)$$

2.4.2. Hall conductance. By a simple back-of-the-envelope argument, the Hall conductance is related to the edge capacitance, i.e. to the charge ΔQ that is accumulated on a given edge in response to an applied voltage V. One quickly derives the following expression for the Hall conductance:

$$G \Big/ \left[\frac{e^2}{h}\right] = \frac{1}{eV}\left[-\int_0^\infty d\epsilon\, \bar{n}_p(\epsilon + eV) + \frac{1}{p}\int_0^\infty d\epsilon\, \bar{n}_{1/p}\left(\epsilon - \frac{e}{p}V\right)\right]. \tag{2.13}$$

Using the properties of the distribution functions, one shows that this expression is independent of the temperature, and gives $G = \frac{1}{p}\frac{e^2}{h}$ for the $\nu = \frac{1}{p}$ edge. For $T = 0$, equation (2.13) reduces to

$$G = \bar{n}_g^{\max}\frac{q^2}{h} \tag{2.14}$$

with q the charge and g the statistics parameter of the quasi-particles that are pulled into the edge by the applied voltage. Depending on the sign of V these are the edge electrons ($q = -e$, $\bar{n}^{\max} = \frac{1}{p}$) or the quasi-holes ($q = \frac{e}{p}$, $\bar{n}^{\max} = p$); both give the canonical value of the Hall conductance.

3. CS models and fractional statistics

3.1. Inverse square exchange in the CFT setting

While the states specified in (2.3) form a complete basis of the chiral Hilbert space, they are not mutually orthogonal, and as such they do not form a proper starting point for further analysis. In principle, one may go through an orthogonalization procedure to arrive at a proper canonical quasi-particle basis. Here, we shall reach this goal in a more efficient way, by exploiting the close connection with so-called Calogero–Sutherland (CS) models of many-body quantum mechanics.

The CS model describes the (non-relativistic) quantum mechanics of N particles on a circle, with a two-body interaction that is proportional to the inverse square of the chord distance between the particles. In [9], Iso demonstrated that the limit $N \to \infty$ of a CS model with interaction strength $p(p-1)$ can be identified with the $c = 1$ CFT of a chiral boson compactified on a circle with radius $R^2 = p$. Iso also specified a collective Hamiltonian H_{CS}, acting in the CFT Hilbert space, whose eigenstates precisely correspond to the multi-particle states of the underlying CS model.

It turns out that the eigenstates of H_{CS} are in one-to-one correspondence with the states specified in equation (2.3): by adding subleading terms to the expressions in (2.3), one arrives at a (orthogonal and complete) set of eigenstates of H_{CS}. Comparing with Iso's formulation, one finds that the 'superfermions' of [9] correspond to what we call edge electrons and the 'anyons' of [9] are the edge quasi-holes of the fqH system. Nevertheless, the 'CS basis' specified by Iso differs from the 'fqH basis' of this paper through the way in which the quasi-particle content of a given state is specified. In subsection 3.4 below we shall spell out the precise correspondence between the two formulations.

The spectrum of the CS models has been analysed with the help of so-called Jack symmetric polynomials, which provide explicit wavefunctions and eigenstates for the CS Hamiltonian. With the help of 'Jack technology', important conjectures [19] concerning zero-temperature correlation functions of models with inverse square exchange have been proven [20, 21]. In the present paper, where we are interested in finite-temperature correlation functions, we shall apply the 'Jack technology' to obtain a set of selection rules for form factors that are relevant

to computations at finite temperature. We shall complement these considerations with explicit computations of form factors involving states with up to two quasi-particles.

3.2. The Hamiltonian H_{CS} and the fqH basis

To specify the operator H_{CS}, we employ a free chiral scalar field $\varphi(z)$. In terms of this scalar field, the charged fields J and ϕ take the form of so-called vertex operators,

$$J(z) = e^{-i\sqrt{p}\varphi}(z) \qquad \phi(z) = e^{i\frac{1}{\sqrt{p}}\varphi}(z). \tag{3.1}$$

The operator $Q = \oint(i\sqrt{p}\partial\varphi)$ measures the electric charge in units of $\frac{e}{p}$. Following [9], we define

$$H_{CS} = \frac{p-1}{p}\sum_{l=0}^{\infty}(l+1)(i\sqrt{p}\partial\varphi)_{-l-1}(i\sqrt{p}\partial\varphi)_{l+1} + \frac{1}{3p}\left[(i\sqrt{p}\partial\varphi)^3\right]_0 \tag{3.2}$$

where $\partial\varphi(z) = \sum_l(\partial\varphi)_l z^{-l-1}$ and where the second term on the right-hand side denotes the zero-mode of the normal ordered product of three factors $(i\sqrt{p}\partial\varphi)(z)$. As a first result, one finds the following action of H_{CS} on states containing a single quasi-particle of charge $Q = +1$ or $Q = -p$:

$$H_{CS}\,\phi_{-1/2p-n}|0\rangle = h_\phi(n)\,\phi_{-1/2p-n}|0\rangle \qquad h_\phi(n) = \left[\frac{1}{3p} + pn\left(n+\frac{1}{p}\right)\right]$$

$$H_{CS}\,J_{-p/2-m}|0\rangle = h_J(m)\,J_{-p/2-m}|0\rangle \qquad h_J(m) = \left[-\frac{p^2}{3} - m(p+m)\right]. \tag{3.3}$$

We would like to stress that the fact that both J_s and ϕ_t diagonalize H_{CS} is quite non-trivial. If one evaluates H_{CS} on any vertex operator $\phi^{(Q)}$ (of charge $Q\frac{e}{p}$), one typically runs into the field product $(T\phi^{(Q)})(z)$, where $T(z) = -\frac{1}{2}(\partial\varphi)^2(z)$ is the stress-energy of the scalar field φ. Only for $Q = 1$ and $-p$ do such terms cancel and do we find that the quasi-particle states are eigenstates of H_{CS}.

We can now continue and construct eigenstates of H_{CS} which contain several ϕ- or J-quanta. What one then finds is that the simple product states specified in (2.3) are not H_{CS} eigenstates, but that they rather act as head states that need to be supplemented by a tail of subleading terms. For the H_{CS} eigenstate headed by the multi-particle state (2.3) (with unit coefficient), we shall use the notation

$$|\{m_j; n_i\}\rangle^Q \tag{3.4}$$

so that

$$H_{CS}|\{m_j; n_i\}\rangle^Q = \left[\frac{Q^3}{3p} + \sum_{j=1}^{M} h_J((j-1)p - Q + m_j)\right.$$

$$\left. + \sum_{i=1}^{N} h_\phi\left(\frac{1}{p}(Q+i-1) + n_i\right)\right]|\{m_j; n_i\}\rangle^Q. \tag{3.5}$$

The states (3.4), with m_j, n_i and Q as specified in and below (2.3), form a complete and orthogonal basis for the chiral Hilbert space.

For the sake of illustration, we give explicit results for a few simple states of the fqH basis. The one-particle states over the $Q = 0$ vacuum are given by

$$|\{m_1\}\rangle^0 = J_{-p/2-m_1}|0\rangle \qquad |\{n_1\}\rangle^0 = \phi_{-1/2p-n_1}|0\rangle. \tag{3.6}$$

The norms of these states can be evaluated explicitly by exploiting the (generalized) commutation relations satisfied by the modes J_s and ϕ_t. One finds

$$^0\langle\{m_1\}|\{m_1\}\rangle^0 = C_{m_1}^{(-p)} \qquad ^0\langle\{n_1\}|\{n_1\}\rangle^0 = C_{n_1}^{(-1/p)} \qquad (3.7)$$

where $C_k^{(\alpha)}$ are the expansion coefficients of $(1-x)^\alpha = \sum_{k\geqslant 0} C_k^{(\alpha)} x^k$.

In our discussion below we shall often restrict ourselves to the vacuum sector $Q = 0$, and omit the explicit sector label Q on the fqH states (3.4).

For later use, we present the explicit form of the fqH states $|\{m_2, m_1\}\rangle$ and $|\{m_1; n_1\}\rangle$ at $p = 2$

$$|\{m_2, m_1\}\rangle = |m_2, m_1\rangle + \frac{2}{m_2 - m_1 + 3} \sum_{l>0} |m_2 + l, m_1 - l\rangle \qquad (3.8)$$

$$|\{m_1; n_1\}\rangle = |m_1; n_1\rangle - \frac{1}{(m_1 + 2n_1 + 1)} \sum_{l>0} |m_1 + l; n_1 - l\rangle \qquad (3.9)$$

with

$$|m_2, m_1\rangle = J_{-3-m_2} J_{-1-m_1} |0\rangle \qquad |m_1; n_1\rangle = J_{-1-m_1} \phi_{-\frac{1}{4}-n_1} |0\rangle. \qquad (3.10)$$

By explicit evaluation, we obtain the following norms for these states (again at $p = 2$):

$$N_{\{m_2,m_1\}} = \langle\{m_2, m_1\}|\{m_2, m_1\}\rangle = \frac{m_2 - m_1 + 1}{m_2 - m_1 + 3}(m_2 + 3)(m_1 + 1)$$

$$N_{\{m_1;n_1\}} = \langle\{m_1; n_1\}|\{m_1; n_1\}\rangle = \frac{m_1 + 2n_1 + 2}{m_1 + 2n_1 + 1}(m_1 + 1)C_{n_1}^{(-1/2)}. \qquad (3.11)$$

3.3. Jack polynomials and the CS bases

In the previous section, we specified a complete set of eigenstates of the Hamiltonian H_{CS} in terms of charged quasi-particles J and ϕ. It is clear that these same eigenstates can be obtained by applying an appropriate polynomial in the modes $a_n = (\partial\varphi)_n$ of the auxiliary scalar field to a vacuum state $|q\rangle$. It turns out that the polynomials that are needed are so-called Jack polynomials. In appendix B we briefly discuss some of their relevant properties.

Following Iso [9], we may specify a basis of CS eigenstates as follows:

$$|\{\mu\}_J, q\rangle = J_{\{\mu'\}}^{1/p}(\{\sqrt{p}a_{-n}\})|q\rangle = J_{\{\mu'\}}^{1/p}|q\rangle \qquad (3.12)$$

with the $U(1)$ charge q running over all integers, and $\{\mu\}$ running over all Young tableaux. The norms of these states are given by

$$\langle\{\mu\}_J, q|\{\mu\}_J, q\rangle = j_{\mu'}^{1/p}. \qquad (3.13)$$

An alternative 'dual' basis, consists of the states

$$|\{\nu\}_\phi, q\rangle = J_{\{\nu'\}}^{p}\left(\left\{\frac{a_{-n}}{\sqrt{p}}\right\}\right)|q\rangle = J_{\{\nu'\}}^{p}|q\rangle \qquad (3.14)$$

with q integer and $\{\nu\}$ running over all Young tableaux, with norms given by

$$\langle\{\nu\}_\phi, q|\{\nu\}_\phi, q\rangle = j_{\nu'}^{p}. \qquad (3.15)$$

3.4. Correspondence between fqH and CS bases

Knowing that both the fqH quasi-particle basis and the CS Jack polynomial basis (3.12) are complete bases of orthogonal eigenstates of the operator H_{CS}, one quickly concludes that there is a one-to-one identification between these two bases. In this section we explicitly describe this one-to-one correspondence.

We start by observing that the Jack state

$$|\{\mu\}_J, q\rangle = J_{\{\mu'\}}^{1/p}|q\rangle \tag{3.16}$$

can be written as the sum of a leading state

$$J_{-m_{\tilde{M}}+q+p/2}J_{-m_{\tilde{M}-1}+q+3p/2}\ldots J_{-m_1+q+(2\tilde{M}-1)p/2}|q+\tilde{M}p\rangle \tag{3.17}$$

and a 'tail' of sub-leading corrections, where 'subleading' refers to the triangular form of H_{CS} on states of the form (3.17). Here we identified $m_j = \mu_{\tilde{M}+1-j} \geqslant 1$, with $\tilde{M} = l(\{\mu\})$.

Similarly, we identify

$$|\{\nu\}_\phi, q\rangle = J_{\{\nu'\}}^p|q\rangle \tag{3.18}$$

with the H_{CS} eigenstate headed by

$$\phi_{-n_{\tilde{N}}-q/p+1/2p}\phi_{-n_{\tilde{N}-1}-q/p+3/2p}\ldots\phi_{-n_1-q/p+(2\tilde{N}-1)/2p}|q-\tilde{N}\rangle \tag{3.19}$$

with $n_i = \nu_{\tilde{N}+1-i} \geqslant 1$ and $\tilde{N} = l(\{\nu\})$.

Note that, as they stand, expressions (3.17) and (3.19) are, in general, not of the form (2.3) that defines a member of the fqH basis.

Using the above, we find the following identifications for fqH states which contain only one type of mode operator:

$$\begin{aligned}|\{n_i\}\rangle^Q &= |\{\nu\}_\phi, Q+N\rangle\\|\{m_j\}\rangle^Q &= |\{\mu\}_J, Q-pM\rangle\end{aligned} \tag{3.20}$$

with $\nu_i = n_{N-i+1} \geqslant 1$ for $\tilde{N}-i \geqslant 0$ and similarly $\mu_j = m_{M-j+1} \geqslant 1$ for $\tilde{M}-j \geqslant 0$. Note that only the n_i that are non-zero become a part of the tableau $\{\nu\}$; the ϕ modes with $n_i = 0$ change the charge of the vacuum without exciting any of the $(\partial\varphi)_n$ modes. (A similar remark applies to the J modes with $m_j = 0$.)

The duality equation (B.3) on the Jack polynomials leads to the following duality relation for the Jack operators:

$$J_{\{\lambda'\}}^p = (-1)^{|\lambda|}j_{\{\lambda\}}^p J_{\{\lambda\}}^{1/p}. \tag{3.21}$$

This relation enables us rewrite the states $|\{n_i\}\rangle^Q$ to either the $|\{\nu'\}_J, Q+N\rangle$ or the $|\{\nu\}_\phi, Q+N\rangle$ form. We can, for example, rewrite

$$\begin{aligned}|\{n_i\}\rangle^Q &= |\{\nu\}_\phi, Q+N\rangle\\&= (-1)^{|\nu|}j_{\{\nu'\}}^p|\{\nu'\}_J, Q+N\rangle.\end{aligned} \tag{3.22}$$

The last state is equivalent to the state which has

$$J_{-\nu'_1+Q+N+p/2}J_{-\nu'_2+Q+N+3p/2}\ldots J_{-\nu'_{n_N}+Q+N+(2n_N-1)p/2}|Q+N+n_N p\rangle \tag{3.23}$$

as its leading state (assuming $n_1 > 0$).

One may explicitly check that the eigenvalues of both the Virasoro zero-mode L_0 and the CS Hamiltonian H_{CS} are invariant under the duality transformation that identifies the state headed by (3.19) to the state headed by (3.23).
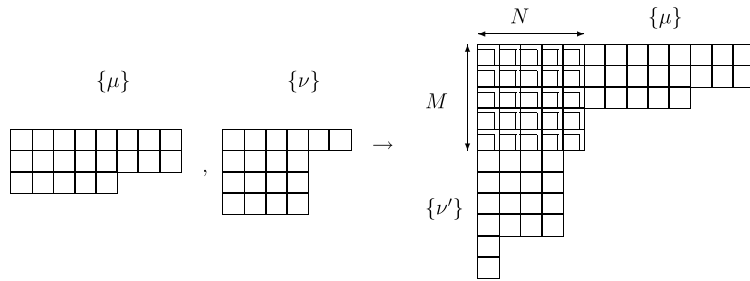
Figure 1. Mapping from the state $|(8, 8, 5, 0, 0), (6, 4, 4, 4, 0)\rangle^0$ in the fqH basis to the state $|(13, 13, 10, 5, 5, 4, 4, 4, 4, 1, 1)_J, q = 5 - 5p\rangle$ in the CS basis.

If there are no J-operators present in the fqH head state we can use the duality to transform the ϕ-operators in the head state into J-operators and we achieve our goal of identifying the fqH state with a member of the CS J-basis. If the fqH state has both J- and ϕ-operators present, we can still map the ϕ-operators to a dual set of J-operators. Starting from the state $|\{m_j\}, \{n_i\}\rangle^Q$ we see that, upon using the identity (3.22), the J modes associated with the m_j 'see' their vacuum shifted from $|Q\rangle$ to $|Q + N\rangle$. Since the vacuum charge leads to a shift in the values of the mode-indices (see (3.17)), this means that in the CS basis, the corresponding J modes will be labelled by $m_i + N$ instead of m_i. It is important to remark that this shift does not affect the contribution of the J modes to the eigenvalue of H_{CS} on the state. This is because the eigenvalues of H_{CS}, as specified in equation (3.5), depend directly on the full mode-indices in the head state and not just on the labels m_j, as can be seen by comparing equation (3.5) with (2.3).

We can now give the full mapping from a fqH basis state to a state in the CS basis (see figure 1 for an explicit example)

$$|\{m_j; n_i\}\rangle^Q = |\{\sigma\}_J, Q + N - pM\rangle \qquad \text{with} \quad \{\sigma\} = (\{m\} + N^M) \cup \{\nu'\} \tag{3.24}$$

where the sum of the partitions is $\{\mu\} + N^M = (\mu_1 + N, \mu_2 + N, \ldots, \mu_M + N)$, and the cup product $\{\lambda\} \cup \{\rho\}$ denotes the partition obtained from sorting the parts $(\lambda_1, \ldots, \lambda_S, \rho_1, \ldots, \rho_R)$ in descending order.

The mapping from the CS basis back to the fqH basis is slightly more complicated. We start from a CS state $|\{\sigma\}_J, q\rangle$, with $S = l(\{\sigma\})$, to which we associate a multi-J state as in (3.17). We note that the quantity $\sigma_j - q - pj$ decreases with increasing label j which allows us to fix j such that

$$\sigma_j - q - pj \geqslant 0 \qquad \sigma_{j+1} - q - p(j + 1) < 0. \tag{3.25}$$

The condition on σ_j guarantees that the associated J mode is an allowed mode on a fqH vacuum $|Q\rangle$ with $-(p - 1) \leqslant Q \leqslant 0$. We now consider the state that remains when all J modes left from and at the position j are removed. With the use of duality this state can be rewritten as

$$|\{\nu'\}_J, q + pj\rangle \propto |\{\nu\}_\phi, q + pj\rangle \tag{3.26}$$

with

$$\nu'_i = \sigma_{i+j} \qquad \text{for} \quad i = 1, \ldots, S - j. \tag{3.27}$$

The ϕ-operators in the leading state of $|\{\nu\}_\phi, q + pj\rangle$ act on the vacuum with charge $\tilde{Q} = q + pj - \sigma_{j+1}$ vacuum. The second condition in (3.25) guarantees that $\tilde{Q} \geqslant -(p - 1)$.

If one has $\tilde{Q} \leqslant 0$, one identifies $Q = \tilde{Q}$ as the vacuum charge of the fqH basis state. If, however, $\tilde{Q}$ is larger than zero the state $|\tilde{Q}\rangle$ was created from $|0\rangle$ using ϕ-operators with the highest allowed mode index. From this argument we obtain that the state $|\{\sigma\}_J, q\rangle$ from the CS basis can be rewritten as $|\{m_i; n_i\}\rangle^Q$ with the following rules for selecting Q, M and N and the mode indices $\{m_i\}$ and $\{n_i\}$:

$$M = j \qquad N = \max(q + pj, \sigma_{j+1}) \qquad Q = q + pM - N \qquad (3.28)$$

$$
\begin{aligned}
m_i &= \sigma_{j+1-i} - N & \text{for} \quad i &= 1, \ldots, j \\
n_i &= v_{\sigma_{j+1}+1-i} & \text{for} \quad i &= 1, \ldots, \sigma_{j+1} \\
&= 0 & \text{for} \quad i &= \sigma_{j+1} + 1, \ldots, N
\end{aligned}
\qquad (3.29)
$$

with j and $\{v\}$ as specified in (3.25) and (3.27).

3.5. Norms for the fqH basis

Of importance for later calculations are the norms of the states $|\{m_j; n_i\}\rangle^Q$ of the fqH basis. These norms can be evaluated by using the fqH–CS correspondence, together with the norms in the CS basis, as specified in (3.13) and (3.15). The general result is

$$N_{\{m_j;n_i\}} = \left| |\{m_j; n_i\}\rangle^Q \right|^2 = (j^p_{\{v'\}})^2 \, j^{1/p}_{\{\sigma'\}} \qquad (3.30)$$

where $\{v\}$ and $\{\mu\}$ are the tableaux corresponding to $\{n_i\}$ and $\{m_j\}$, respectively, and the tableau $\{\sigma\}$ is specified in (3.24). The norm can be factorized into the norms of the J and ϕ parts separately times an extra factor associated with the added partition N^M

$$N_{\{m_j;n_i\}} = j^p_{\{v'\}} j^{1/p}_{\{\mu'\}} \prod_{(i,j)\in M^N} \frac{(\mu_j - i + M) + p(v_i - j + N + 1)}{(\mu_j - i + 1 + M) + p(v_i - j + N)}. \qquad (3.31)$$

The expressions (3.11) are special cases of this general formula.

In the case where $m_j, n_i \gg 1$ the expressions simplify and one finds the following factorized form:

$$N_{\{m_j;n_i\}} \approx \prod_{j=1}^{M} \frac{m_j^{p-1}}{\Gamma(p)} \prod_{i=1}^{N} \frac{n_i^{1/p-1}}{\Gamma(1/p)}. \qquad (3.32)$$

4. Form factors

4.1. Vacuum form factors

We start by considering the simplest non-vanishing form factors of the basic electron operators $J(z)$, $J^\dagger(z)$ versus the multi-particle states in the fqH basis

$$
\begin{aligned}
\langle 0 | J_{p/2+m} | \{n_p, \ldots, n_2, n_1\} \rangle_N &= [N_{\{n_p,\ldots,n_2,n_1\}}]^{1/2} f_J(n_p, \ldots, n_1) \, \delta_{m,n_p+\cdots+n_1} \\
\langle 0 | J^\dagger_{+p/2+m} | \{m_1\} \rangle_N &= [N_{\{m_1\}}]^{1/2} f_{J^\dagger}(m_1) \, \delta_{m,m_1}
\end{aligned}
\qquad (4.1)
$$

where the subscript N indicates that the state has been properly normalized.

One immediately finds

$$f_{J^\dagger}(m_1) = 1. \qquad (4.2)$$

We briefly explain the exact evaluation of the form factor $f_J(n_p, \ldots, n_1)$ as defined in (4.1). Let us consider the special case $p = 2$ first. In that case the operator $J(z)$ has conformal

dimension one and may be identified with one of the currents of the affine Kac–Moody algebra $\widehat{su(2)}_1$ (see appendix A). By exploiting the operator product expansion

$$\phi(w_1)\phi(w_2) = (w_1 - w_2)^{+1/2}\left[J^\dagger(w_2) + \mathcal{O}(w_1 - w_2)\right] \tag{4.3}$$

one obtains

$$J^\dagger(w_2) = \oint_{C_{w_2}} \frac{\mathrm{d}w_1}{2\pi\mathrm{i}}(w_1 - w_2)^{-3/2}\phi(w_1)\phi(w_2). \tag{4.4}$$

Using the expansion formula (B.7) we obtain

$$J^\dagger(w_2)|0\rangle = \sum_{n_2,n_1} P^{1/2}_{\{n_2,n_1\}}(w_2, w_2)|\{n_2, n_1\}_\phi, q = 2\rangle \tag{4.5}$$

and it follows that

$$\langle 0|J_{1+m}|\{n_2, n_1\}\rangle_N = [N_{\{n_2,n_1\}}]^{1/2}\, P^{1/2}_{\{n_2,n_1\}}(1, 1)\,\delta_{m,n_2+n_1}. \tag{4.6}$$

For general p one obtains a similar result in terms of Jack polynomials with label $\frac{1}{p}$. Using explicit expressions [22, 21] for $P^{1/p}_{\{\ldots\}}(1, \ldots, 1)$, we obtain

$$f_J(n_p, \ldots, n_1) = \prod_{i=0}^{p-1}\frac{\Gamma(1/p)}{\Gamma(1 - i/p)}\prod_{i<j}\frac{\Gamma(n_j - n_i + (j - i + 1)/p)}{\Gamma(n_j - n_i + (j - i)/p)}. \tag{4.7}$$

In the large volume limit, where all $n_i \gg 1$, this becomes

$$f_J(n_p, \ldots, n_1) = \frac{[\Gamma(1/p)]^p}{\prod_{i=0}^{p-1}\Gamma(1 - i/p)}\prod_{i<j}(n_j - n_i)^{\frac{1}{p}}. \tag{4.8}$$

The form (4.8) of the form factor can be viewed as a limit in (chiral) CFT of a result on correlation functions for the 'classical' model of quantum mechanics with inverse square exchange. This result was conjectured by Haldane [19] and later proven in [20, 21].

An important insight is that there are no other non-vanishing form factors of the state $J^\dagger_{-p/2-m}|0\rangle$ with the elements of the fqH basis. In other words, the spectral weight of this state is completely accounted for by states having precisely the minimal number of p quasi-holes. On the basis of this observation, it has been proposed that the $T = 0$ particle system underlying the fqH basis be viewed as an 'ideal gas of fractional statistics particles'. In a formula, this completeness is expressed by the following identity for the $T = 0$ two-point function of the edge electron operator J_{-t}

$$\frac{1}{m+1}\langle 0|J_{m+1}J^\dagger_{-m-1}|0\rangle = \frac{1}{m+1}\sum_{n_2\geqslant n_1, n_2+n_1=m}\langle 0|J_{m+1}|\{n_2, n_1\}\rangle_{NN}\langle\{n_2, n_1\}|J^\dagger_{-m-1}|0\rangle$$

$$= \frac{1}{m+1}\sum_{n_2\geqslant n_1, n_2+n_1=m}\frac{(n_2 - n_1)}{(n_2 n_1)^{1/2}} = \int_0^{\frac{1}{2}}\mathrm{d}x\frac{(1 - 2x)}{[x(1 - x)]^{1/2}} = 1 \tag{4.9}$$

where $x = \frac{m_1}{m}$ and we inserted asymptotic expressions valid for $m, n_2, n_1 \gg 1$. This result is in agreement with a direct computation using algebraic properties of the J modes.

We remark here that, as we shall see in the following sections, the structure of more general form factors, shows 'many-body effects' and is not easily reconciled with a notion of an ideal gas of fractional statistics particles.

4.2. More general form factors

We now consider more general form factors for the edge electron 'annihilation operator' $J^\dagger(z)$. The simplest form factor with a two-particle in-state is

$$_N\langle\{m_1'\}|J^\dagger_{3p/2+m}|\{m_2,m_1\}\rangle_N = \left[\frac{N_{\{m_2,m_1\}}}{N_{m_1'}}\right]^{1/2} f_{J|JJ}(m_1',m_2,m_1)\,\delta_{m,m_2+m_1-m_1'}. \tag{4.10}$$

Using the expansion formula (B.8) we derive the following general result:

$$f_{J|JJ}(m_1',m_2,m_1) = \sum_{s=0}^{p} C_s^{(p)} p_{m_1'-m_1-s}^{(p)} \tag{4.11}$$

where

$$P_{\{m_2,m_1\}}^p(z_1,z_2) = z_1^{m_2} z_2^{m_1} \sum_{l=0}^{m_2-m_1} p_l^{(p)}\left(\frac{z_2}{z_1}\right)^l. \tag{4.12}$$

Specializing to $p = 1, 2, 3$, we have the following explicit results

$p = 1$: $f_{J|JJ}(m_1',m_2,m_1) = \delta_{m_1'=m_1} - \delta_{m_1'=m_2+1}$

$p = 2$: $f_{J|JJ}(m_1',m_2,m_1) = \delta_{m_1'=m_1} + \delta_{m_1'=m_2+2}$

$$-\frac{2}{m_2-m_1+1}\Theta(m_1 < m_1' < m_2+2) \tag{4.13}$$

$p = 3$: $f_{J|JJ}(m_1',m_2,m_1) = \delta_{m_1'=m_1} - \delta_{m_1'=m_2+3}$

$$-\frac{6(m_2+m_1-2m_1')}{(m_2-m_1+1)(m_2-m_1+2)}\Theta(m_1 < m_1' < m_2+3).$$

We also consider the case where the in-state contains one ϕ and one J quantum

$$_N\langle\{n_1'\}|J^\dagger_{p/2+m}|\{m_1;n_1\}\rangle_N = \left[\frac{N_{\{m_1;n_1\}}}{N_{n_1'}}\right]^{1/2} f_{\phi|J\phi}(n_1',m_1,n_1)\,\delta_{m,m_1+n_1-n_1'}. \tag{4.14}$$

Using the expansion formula

$$: J(z)\phi(w) : |0\rangle = \sum_{m_1;n_1} r_{m_1,n_1}(z,w)|\{m_1;n_1\}\rangle \tag{4.15}$$

with

$$r_{m,n}(z,w) = z^{m+1}w^n - \frac{m+1+pn}{m+p(n+1)}z^m w^{n+1} \tag{4.16}$$

we derive

$$f_{\phi|J\phi}(n_1',m_1,n_1) = \left[\delta_{n_1'=n_1} - \frac{p-1}{m_1+p(n_1+1)}\Theta(n_1' > n_1)\right]. \tag{4.17}$$

While the results for $p = 1$ are a direct consequence of the Wick theorem, the expressions for $p \neq 1$ show that the 'ideal gas interpretation' is no longer applicable for general p: both form factors (4.10) and (4.14) can be non-vanishing when the energy of the electron annihilation operator $J^\dagger$ does not match the incoming electron energies m_2 or m_1, and where the energy difference is transferred to a second 'spectator particle'. Furthermore, $f_{J|JJ}$ and $f_{\phi|J\phi}$ are not the only non-vanishing form factors of $J^\dagger$ with two incoming particles. For example, there are non-vanishing overlaps between a state created by applying $J^\dagger$ on the two-electron state and states containing more quasi-particles than just a single electron. The additional quasi-particles can be visualized as (neutral) density waves or excitons, which are composed of a single electron and p quasi-holes. In the next subsection, we explore the selection rules that determine in a more general setting the possible out-states for which the form factor of $J^\dagger$ with a given in-state is non-vanishing.

4.3. A form factor selection rule

In this section we put a bound on the possible out-state that arise upon acting with an electron creation or annihilation operator on a given in-state. We perform this analysis in the CS basis, where the systematics of Jack polynomials come to our aid. Using the mapping of section 3.4, the results can be translated to the fqH basis.

We focus on the form factors

$$\langle \{\mu\}_J, q | J_{-m+q+p-p/2} | \{\nu\}_J, q + p \rangle \tag{4.18}$$

with (unnormalized) Jack states as in (3.12).

The power of Jack polynomial technology in analysing this form factor comes from the fact that a product of vertex operators can be written as a sum over products of 'coordinates' Jack polynomials $P^p_{\{\lambda\}}(\{z_i\})$ and 'bosonic modes' Jack operators $J^{1/p}_{\{\lambda\}}$. Both the coordinate and the bosonic mode Jack polynomials can be manipulated using the results for Jack polynomials in the mathematical literature [22, 23]. In appendix B, we used this to rearrange the part of the vertex operators that survives after applying them to the vacuum. Here we apply the same method which enables us to analyse the action of a single-mode operator $J_{-m+q+p-p/2}$ on a state created by a Jack operator. We note that for $m \geqslant 0$ this mode operator creates an additional edge electron.

A product of $N + 1$ edge electron operators acting on a vacuum $|\tilde{q}\rangle$ can be expanded in terms of Jack polynomials and Jack operators,

$$J(w) \prod_{i=1}^{N} J(z_i)|\tilde{q}\rangle = w^{-\tilde{q}} \prod_{i=1}^{N}(w - z_i)^p \prod_{i<j}(z_i - z_j)^p \prod_{i=1}^{N} z_i^{-\tilde{q}}$$

$$\times \left(\sum_{\{\lambda\}} (-1)^{|\lambda|} P^p_{\{\lambda\}}(w) J^{1/p}_{\{\lambda'\}} \right) \left(\sum_{\{\rho\}} (-1)^{|\rho|} P^p_{\{\rho\}}(\{z_i\}) J^{1/p}_{\{\rho'\}} \right) |\tilde{q} - pN - p\rangle. \tag{4.19}$$

The state $|\{\nu\}_J, \tilde{q} - Np\rangle$ with $N = l(\{\nu\})$ can be extracted from $\prod_{i=1}^{N} J(z_i)|\tilde{q}\rangle$ by applying the operator

$$O^p_{\{\nu\},\tilde{q}}(\{z_i\}) = (p^p_{\{\nu\},l(\{\nu\})})^{-1} \left(\prod_{i=1}^{l(\{\nu\})} \oint \frac{dz_i}{2\pi i} \frac{1}{z_i^{-\tilde{q}+1}} \right) P^p_{\{\nu\}}(\{z_i^{-1}\}) \Delta^p(\{z_i^{-1}\}) \tag{4.20}$$

where we have made use of the inner product (B.6) on coordinate-dependent Jack polynomials. The norm $p^p_{\{\nu\},l(\{n\}}$ of the Jack polynomials will drop out of the final result. We can write

$$J_{-m+q+p-p/2}|\{\nu\}_J, \tilde{q} - pl(\{\nu\})\rangle = M_{m-q-p}(w) J(w) O^p_{\{\nu\},\tilde{q}}(\{z_i\}) \prod_{i=1}^{l(\{\nu\})} J(z_i)|\tilde{q}\rangle \tag{4.21}$$

where

$$M_m(w) = \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}}. \tag{4.22}$$

From the definition it is clear that $O^p_{\{\nu\},\tilde{q}}(\{z_i\})$ commutes with $J(w)$, which allows us to interchange the order in the expression above and to use the expansion (4.19) of $J(w) \prod_{i=1}^{N} J(z_i)|\tilde{q}\rangle$ in terms of Jack polynomials. By taking the inner product of the resulting expression with $\langle \{\mu\}_J, q|$, where $q = \tilde{q} - (N + 1)p$, we obtain an expression for the form factor (4.18).

To decide for which choices of $\{\mu\}$, $\{\nu\}$ and m the form factor can be non-vanishing we proceed as follows. We first rewrite the product

$$\prod_{i=1}^{N}(w - z_i)^p = \sum_{\{n_i\}} C_{\{n_i\}}^{(p)} w^{pN} \prod_{i=1}^{N}\left(\frac{z_i}{w}\right)^{n_i} \tag{4.23}$$

where $n_i = 0, 1, \ldots, p$. We insert this into the expansion given in (4.19), and we use the fact that $P_{\{\lambda\}}^p(w)$ is zero when $\{\lambda\}$ is not of the form $\{\lambda\} = (\lambda_1, 0, 0, \ldots)$. The contour integration contained in $M_{m-p-q}(w)$ selects the following value for λ_1:

$$\lambda_1 = |n| + m. \tag{4.24}$$

Varying $|n| = \sum_i n_i$ over all allowed values $|n| = 0, \ldots, pN$, we find that λ_1 has to satisfy the inequalities

$$m \leqslant \lambda_1 \leqslant m + pN. \tag{4.25}$$

We write $\langle\{\mu\}_J | \{\lambda\}_J \{\rho\}_J\rangle$ for the inner product $\langle\{\mu\}_J, q| J_{\{\lambda'\}}^{1/p} J_{\{\rho'\}}^{1/p} | q\rangle$. Using a result by Stanley (proposition 5.3 from [22]) we learn that this inner product is non-zero if and only if $\rho \subseteq \mu$ and μ/ρ is a horizontal λ_1-strip. The relation $\rho \subseteq \mu$ indicates that for all i we have $\rho_i \leqslant \mu_i$. The skew tableau μ/ρ is the tableau containing all boxes which are in the tableau μ but not in the tableau ρ. If every column of the skew tableau contains at most one box it is called a horizontal strip and if, furthermore, the total number of boxes in it is λ_1 it is called a horizontal λ_1-strip.

A particular consequence is that λ_1 satisfies

$$0 \leqslant \lambda_1 \leqslant \mu_1. \tag{4.26}$$

Combining this inequality with (4.25), we conclude that

$$\max(0, m) \leqslant \lambda_1 \leqslant \mu_1 \tag{4.27}$$

which implies that

$$\mu_1 < m \quad \Rightarrow \quad J_{m-q-p+p/2}^{\dagger} |\{\mu\}_J, q\rangle = 0 \tag{4.28}$$

in agreement with explicit step functions in the form factors (4.13) and (4.17).

The form factor is now written as

$$\langle q, \{\mu\}_J | J_{-m+q+p-p/2} | \{\nu\}_J, q + p\rangle = (p_{\{\nu\},l(\{\nu\})}^{p})^{-1} \sum_{\{n\},\{\lambda_1\},\{\rho\}} \delta_{|n|+m,\lambda_1} C_{\{n\}}^{(p)} \langle\{\mu\}_J | \{\lambda_1\}_J \{\rho\}_J\rangle$$

$$\times \left(\prod_{i=1}^{N} M_0(z_i)\right) m_{\{n\}}(\{z_i\}) \Delta^p\left(\{z_i^{-1}\}\right) \Delta^p(\{z_i\}) P_{\{\rho\}}^p(\{z_i\}) P_{\{\nu\}}^p\left(\{z_i^{-1}\}\right) \tag{4.29}$$

where the summations extend over $i, j = 1, \ldots, N$ and $\{n\} = (n_1, \ldots, n_N)$ with $n_i \in 0, 1, \ldots, p$.

The last part of this expression is the inner product on products of Jack polynomials with a finite number of arguments $\{z_i\}$. In appendix C we discuss restrictions on the tableaux $\{n\}$, $\{\nu\}$ and $\{\rho\}$ that follow from imposing that this final inner product be non-zero. Combining all ingredients, one arrives at the following:

Form factor selection rule. *The form factor*

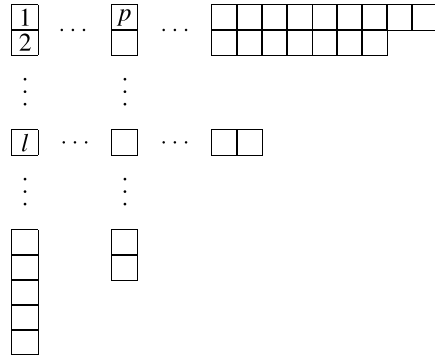$$\langle\{\mu\}_J, q| J_{-m+q+p-p/2} |\{\nu\}_J, q + p\rangle$$

Figure 2. l arms and p legs.

can only be non-zero if $\{\mu\}$, $\{\nu\}$, m satisfy the following conditions:

(a) $|\nu| + m = |\mu|$
(b)   1. $\nu_j \geqslant \mu_{j+1}$ for all j
      2. $\nu_i \leqslant p$ for $i > l(\{\mu\})$
(c) $m + \sum_{i \geqslant l(\{\mu\})} \nu_i \leqslant \mu_1$
(d) $\sum_{i=1}^{j} \nu_i \leqslant \sum_{i=1}^{j} (\mu_i + p)$.

These conditions imply that the tableau $\{\nu\}$ should have at most p legs and $l(\{\mu\})$ arms, see figure 2.

We refer to appendix C for a complete proof of this result.

We remark that the above selection rule can be viewed as a generalization of a selection rule that was used by Lesage *et al* [21] for the evaluation of the zero-temperature density–density correlation function in the Calogero–Sutherland model. These authors found that the (neutral) density operator ρ when acting on the vacuum creates an 'exciton' with p quasi-holes and a single electron, corresponding to a Young tableau with p legs and a single arm. In the processes described by the form factor discussed in this section a similar structure is found. Starting from a multi-J state described by a tableau $\{\mu\}$, the operator labelled by m annihilates one of the J-quanta. If there is a mismatch between the modes of the operator and of the quantum that is annihilated, the remaining momentum is carried away by a density fluctuation, which roughly speaking corresponds to one extra arm and the p legs that can be present in the tableau $\{\nu\}$. If one starts from a state which, in the fqH basis, has a number of J-quanta and N quasi-holes, with maximal mode n_N, one finds that upon annihilating a J mode up to $n_N + 1$ excitons can be created.

4.4. Relation with the S-matrix approach

In this section, we consider the structure of the quasi-particle form factors from the point of view of an associated S-matrix structure.

Via the TBA procedure, the distribution functions for fractional exclusion statistics are linked to an S-matrix with the following dependence on particle rapidities $\theta = \theta_2 - \theta_1$:

$$S_{ab}(\theta) = \exp[2\pi \mathrm{i}(\delta_{ab} - G_{ab})\Theta(\theta)]. \tag{4.30}$$

Although the quasi-particle states that we have considered are part of the discrete spectrum of a finite-size system, it is natural to identify the quasi-particle states with a set of asymptotic

particle states in a scattering theory with two-body S-matrix of this type, with diagonal statistics matrix $G_{11} = p$, $G_{22} = \frac{1}{p}$.

Via the well known form factor axioms, this identification leads to specific properties of the form factors. In particular, we expect factors

$$(\epsilon_i - \epsilon_j)^p \qquad (\tilde{\epsilon}_i - \tilde{\epsilon}_j)^{1/p} \tag{4.31}$$

in form factors with particles $J(\epsilon_i)$ and $\phi(\tilde{\epsilon}_j)$ in the in-state, and annihilation poles between particles in the in- and out-states.

The explicit result (4.8) for the vacuum form factor f_J has the expected zeros $(n_i - n_j)^{1/p}$. For the more general form factors discussed in section 4.2 the structure is less clear. We observe, however, that, upon heuristically replacing

$$\delta_{m_2, m_1} \rightarrow \frac{1}{(\epsilon_2 - \epsilon_1)} \qquad \Theta(m_2 - m_1) \rightarrow \log(\epsilon_2 - \epsilon_1) \tag{4.32}$$

we have (for $p = 1, 2, 3$)

$$[\partial_{\epsilon_1'}]^{p-1} \left[f_{J|JJ}(\epsilon_1', \epsilon_2, \epsilon_1) \right] \rightarrow \frac{(\epsilon_2 - \epsilon_1)^p}{(\epsilon_1' - \epsilon_2)^p (\epsilon_1' - \epsilon_2)^p}. \tag{4.33}$$

It will be most interesting to investigate whether the asymptotic limit of the form factors considered and computed in this paper can be obtained by means of an axiomatic approach.

5. Form factor expansion at finite temperature

5.1. General remarks

In a system of non-interacting electrons, transport properties such as I–V and noise characteristics are obtained by computing the relevant amplitudes for transmission and reflection of single particles, and then performing a statistical average using a one-particle Fermi–Dirac distribution function. An important goal, that we had in mind when setting up the quasi-particle formulation of fqH edges, is to arrive at a similar description of transport processes in these interacting, highly non-Fermi liquid, systems.

As a first attempt in this direction, one may try to simply replace free-electron amplitudes by the corresponding amplitudes for fqH quasi-particles, and simultaneously replace the Fermi–Dirac distribution by an appropriate distribution function for fractional statistics. While, as we shall argue, this idea is essentially correct, we stress that a correct implementation is subtle and involves the important concept of a so-called form factor expansion.

In this section, we shall focus on the following finite-temperature Green functions in the CFT for the $\nu = \frac{1}{p}$ fqH edge:

$$h(\epsilon) = \langle \psi_{\nu=1/p}^{\dagger}(\epsilon)\psi_{\nu=1/p}(\epsilon)\rangle_T \qquad H(\epsilon) = \langle \psi_{\nu=1/p}(\epsilon)\psi_{\nu=1/p}^{\dagger}(\epsilon)\rangle_T \tag{5.1}$$

where the operators $\psi_{\nu=1/p}^{\dagger}(\epsilon)$ and $\psi_{\nu=1/p}(\epsilon)$ are the continuum limits of the edge electron operators J_s and $J_s^{\dagger}$ considered in this paper.

In the next subsection, we recall how this Green function is used for the computation of the I–V characteristics for the tunnelling of electrons into a fqH edge. After that, we give the general form of the form factor expansions for finite-temperature correlation functions. We shall then focus on the case $m = 2$, and explain how the finite-T Green function $h(\epsilon)$ can be approximated in a form factor expansion.

5.2. Kinetic equation for electron tunnelling

As explained in [8, 24], the Green functions (5.1) can be used to computed the finite-temperature I–V characteristics for the tunnelling of electrons from a Fermi liquid (FL) reservoir into a $\nu = \frac{1}{p}$ fqH edge. Starting from the tunnelling Hamiltonian

$$H_{\text{int}} \propto t \int d\epsilon \left[\psi_{\text{FL}}^{\dagger}(\epsilon) \psi_{\nu=\frac{1}{3}}(\epsilon) + \text{h.c.} \right] \tag{5.2}$$

one can show that, in lowest-order perturbation theory, the current–voltage characteristics are given by

$$I(V, T) \propto e\, t^2 \int_{-\infty}^{\infty} d\epsilon \left[f(\epsilon - eV) H(\epsilon) - F(\epsilon - eV) h(\epsilon) \right] \tag{5.3}$$

where $f(\epsilon)$ and $F(\epsilon)$ are the standard Fermi–Dirac distributions for particles and holes. Using the conformal mapping from a plane to a cylinder, or employing an imaginary time approach, one finds the following exact expression for the case $\nu = \frac{1}{3}$:

$$H(\epsilon) = \frac{\epsilon^2 + \pi^2/\beta^2}{e^{-\beta\epsilon} + 1} \qquad h(\epsilon) = \frac{\epsilon^2 + \pi^2/\beta^2}{1 + e^{\beta\epsilon}}. \tag{5.4}$$

They lead to I–V characteristics

$$I(V, T) \propto e\, t^2 \beta^{-3} \left(\frac{\beta eV}{2\pi} + \left(\frac{\beta eV}{2\pi} \right)^3 \right) \tag{5.5}$$

in agreement with the result obtained in different approaches [25, 26]. The I–V characteristics (5.5) show cross-over from a linear (thermal) regime into a power-law behaviour at high voltages and thus presents a clear fingerprint of the Luttinger liquid features of the fqH edge. The experimental results of [27] are in agreement with these predictions. (See [26] for a further theoretical analysis of the data.)

5.3. Form factor expansion

As a prototype study for a form factor expansion based on CFT quasi-particles, we now analyse the Green function $h(\epsilon)$, for $p = 2$ in that spirit. Obviously, an exact result is easily obtained

$$h(\epsilon) = \frac{\epsilon}{e^{\beta\epsilon} - 1}. \tag{5.6}$$

The Bose–Einstein denominator in this expression has its origin in the fact that the operators J, $J^{\dagger}$ satisfy bosonic commutation relations. In the spirit of the quasi-particle formulation of this paper, we wish to treat the J, $J^{\dagger}$-quanta as quasi-particles with exclusion statistics $g = 2$, and see whether we can recover the Green function $h(\epsilon)$ in such an approach.

The Green function $h(\epsilon)$ can be viewed as a one-point function for the operator $N_{\psi}(\epsilon) = \psi_{\nu=1/p}^{\dagger} \psi_{\nu=1/p}(\epsilon)$. In the formulation on the finite system of size L, this operator is represented as $N_J(m) = a J_{-m} J_m^{\dagger}$, with $\epsilon = am$, with $a = \frac{2\pi}{L\rho_0}$ the energy level spacing in the finite-size system. This one-point function is formally expressed as

$$\frac{\sum_{\Psi \in \mathcal{H}} \langle \Psi | N_J(m) | \Psi \rangle}{\sum_{\Psi \in \mathcal{H}} \langle \Psi | \Psi \rangle}. \tag{5.7}$$

The sum runs over a basis the full Hilbert space of the edge CFT, and we can opt for the fqH quasi-particle basis which is discussed in this paper. The idea is now that the matrix

elements $\langle\Psi|N_J(m)|\Psi\rangle$ are dominated by processes where only a few of the quasi-particles that are present in a concrete basis state $|\{m_i; n_j\}\rangle$ participate (we restrict our attention to states in the $Q = 0$ sector of the fqH basis).

For the case at hand, the lowest contributions comes from one-particle states $|\{m_1\}\rangle$, for which one computes the form factor

$$D^{(1,0)}(m; m_1) = {}_N\langle\{m_1\}|J_{-1-m}J^\dagger_{+1+m}|\{m_1\}\rangle_N$$

$$= (m + 1)\delta_{m,m_1} + 2\left(1 - \frac{m+1}{m_1+1}\right)\Theta(m < m_1). \tag{5.8}$$

The expected presence of an edge electron of energy m_1 is given by the distribution function $\bar{n}_2(\epsilon_1 = am_1)$. This leads to the following contribution to the Green function:

$$h^{(1,0)}(\epsilon) = a\sum_{m_1} D^{(1,0)}(m, m_1)\bar{n}_2(am_1). \tag{5.9}$$

If we now consider the form factor of $N_J(m)$ for a two-electron state, we find (see the next subsection) that it is not simply the sum of two one-particle contributions. The leftover part is what we call the irreducible two-electron form factor,

$$D^{(2,0)}(m; m_1, m_2) = {}_N\langle\{m_1, m_2\}|J_{-3-m}J^\dagger_{+3+m}|\{m_2, m_1\}\rangle_N$$

$$- {}_N\langle\{m_1\}|J_{-3-m}J^\dagger_{+3+m}|\{m_1\}\rangle_N - {}_N\langle\{m_2\}|J_{-3-m}J^\dagger_{+3+m}|\{m_2\}\rangle_N. \tag{5.10}$$

It leads to an additional contribution $h^{(2,0)}(m)$ to the Green function

$$h^{(2,0)}(\epsilon) = a\sum_{m_1,m_2} D^{(2,0)}(m; m_1, m_2)\bar{n}_2(am_1)\bar{n}_2(am_2). \tag{5.11}$$

Similarly, we define

$$D^{(1,1)}(m; m_1, n_1) = {}_N\langle\{n_1; m_1\}|J_{-1-m}J^\dagger_{+1+m}|\{m_1; n_1\}\rangle_N - {}_N\langle\{m_1\}|J_{-1-m}J^\dagger_{+1+m}|\{m_1\}\rangle_N \tag{5.12}$$

and

$$h^{(1,1)}(\epsilon) = a\sum_{m_1,n_1} D^{(1,1)}(m; m_1, n_1)\bar{n}_2(am_1)\bar{n}_{1/2}(an_1). \tag{5.13}$$

Continuing in this manner, we build up the following expansion:

$$h = \sum_{M,N} h^{(M,N)}(\epsilon)$$

$$h^{(M,N)}(\epsilon) = a\sum_{\{m_i;n_j\}} D^{(M,N)}(m; \{m_i; n_j\})\prod_i \bar{n}_2(am_i)\prod_j \bar{n}_{1/2}(an_j). \tag{5.14}$$

We remark that an expansion of precisely this type has been proposed by LeClair and Mussardo [11] (see also [12]). This work was done in the context of integrable QFTs, that are fully characterized by a factorized S-matrix. In such a context, the irreducible form factors are constrained by the form factor axioms, and the distribution functions have their origin in a TBA procedure. Although clearly in the same spirit, the analysis that we present here is very different at the technical level. We obtain the relevant form factor by explicit computation in a theory that is regularized by the finite size of the fqH edge, and we have identified the relevant distribution functions by analysing the state counting of the (discrete) spectrum of the finite-size system. We thus do not rely on an underlying (massless) S-matrix point of view. Despite these differences, it seems clear that the two approaches are closely related: in subsection 4.5 we briefly indicated that our form factor has symmetry properties that are expected on the basis of a 'purely statistical S-matrix'. We leave this as an interesting issue for further study.

5.4. Irreducible form factors

To evaluate explicitly the leading terms in the form factor expansion (5.14) for $h(\epsilon)$, we need to evaluate the relevant irreducible form factors. While it is clear that these form factors have very special mathematical properties, we compute them here by a simple brute-force computation, relying on the explicit form of the two-particle states (3.8) and (3.9), and on the algebraic properties of the operators J, $J^\dagger$ and ϕ (see appendix A).

5.4.1. Two electrons. For the irreducible two-electron form factor we find

$$D^{(2,0)}(m; m_2, m_1) = \delta_{m-m_2} \frac{-2(m_2 + 3)}{m_2 - m_1 + 3} + \delta_{m-m_1+2} \frac{-2(m_1 + 1)}{m_2 - m_1 + 1}$$

$$+ \frac{4}{(m_2 - m_1 + 3)} \frac{1}{(m_2 - m_1 + 1)} \frac{1}{(m_1 + 1)(m_2 + 3)}$$

$$\times [\Theta(m < m_1 - 2) \, P(m; m_1, m_2) + \Theta(m < m_2 < m + m_1) \, Q(m; m_1, m_2)$$

$$+ \Theta(m < m_2) \, R(m; m_1, m_2)] \tag{5.15}$$

with

$$P(m; m_1, m_2) = (m_2 - m_1 + 3)(m_1 - m - 2)(2m_1 - m_2 - 3)$$

$$+ (m_1 - m - 2)(m_1 - m - 3)\left(-3m_2 + \tfrac{5}{3}m_1 + \tfrac{1}{3}m - \tfrac{26}{3}\right)$$

$$+ (m + 3)[-2(m_2 - m_1 + 3)(2m_1 - m - 1)$$

$$- 2m_1(m_1 + 1) + (m + 3)(m_2 + m_1 - m + 1)] \tag{5.16}$$

$$Q(m; m_1, m_2) = (m_1 - m_2 + m + 1)\big[(m_2 - m_1 + 3)^2 + 2(m_2 - m_1 + 3)(m_1 - m_2 + m)$$

$$+ \tfrac{2}{3}(m_1 - m_2 + m)(m_1 - m_2 + m - 1)\big]$$

$$R(m; m_1, m_2) = (m_2 - m)(m_1 + 1)(m_2 - m_1 + 3) + \tfrac{1}{3}m_1(m_1 + 1)(m_1 + 3m_2 - 3\,m + 2).$$

The polynomials P, Q and R enjoy special properties, which include

$$(P + Q + R)(m; m_1, m_2) = -\tfrac{1}{3}(m_1 - m_2 - 1)(m_1 - m_2 - 2)(m_1 - m_2 - 3). \tag{5.17}$$

5.4.2. One electron and one quasi-hole. The irreducible form factor with one electron and one hole is found to be

$$D^{(1,1)}(m; m_1, n_1) = \delta_{m_1, m} \frac{m_1 + 1}{m_1 + 2n_1 + 1}$$

$$+ \Theta(m < m_1) \frac{1}{C_{n_1}^{(-1/2)}(m_1 + 2n_1 + 2)(m_1 + 2n_1 + 1)(m_1 + 1)}$$

$$\times \left[C_{n_1 - m_1 + m}^{(-1/2)} \, S(m; m_1, n_1) + C_{n_1}^{(-1/2)} \, T(m; m_1, n_1) \right] \tag{5.18}$$

with

$$S(m; m_1, n_1) = (m_1 + 2n_1 + 1)^2 + (m + n_1 - m_1)\left(\tfrac{8}{3} - 4(m_1 + 2n_1 + 2)\right)$$

$$+ \tfrac{4}{3}(m + n_1 - m_1)^2$$

$$T(m; m_1, n_1) = 2(m_1 - m)((m_1 + 2n_1 + 1)^2 - 1) + 2(2n_1 + 1)(m_1 - m - 1)$$

$$+ 2\left(\tfrac{2}{3}n_1 + 1\right)(2n_1 + 1). \tag{5.19}$$
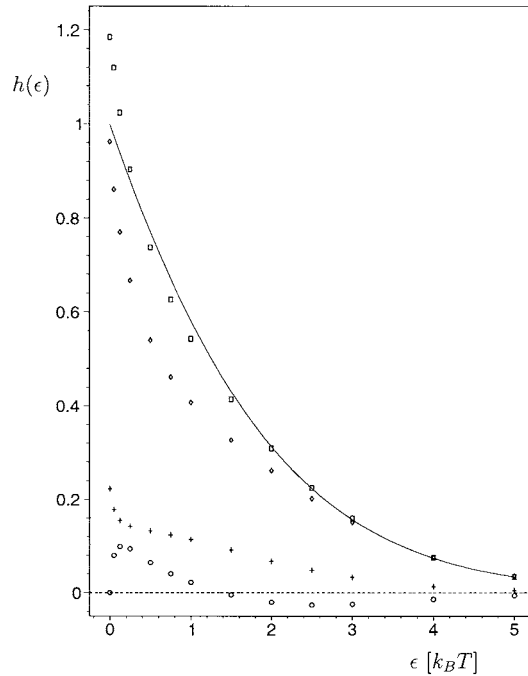
Figure 3. One-particle Green function $h(\epsilon)$ for $p = 2$ as a function of energy. The full curve is the exact result; the data points are the numerical results for: $h^{(1,0)}$ (diamonds), $h^{(2,0)}$ (circles) and $h^{(1,1)}$ (pluses). The sum of all contributions with up to two particles is represented by squares.

5.5. Evaluating the series

With the information collected in the previous subsections, we can evaluate the one- and two-particle contributions $h^{(1,0)}$, $h^{(2,0)}$ and $h^{(1,1)}$ to the Green function $h(\epsilon)$.

Expressions (5.9), (5.11) and (5.13) for $h^{(2,0)}$ and $h^{(1,1)}$ are discrete sums, which we wish to study in the limit $a \to 0$. In this limit, one may view the expressions as Riemann sums and evaluate them using continuous integrals; however, one needs to be careful because the integrands as they stand have singularities, and the sums are not term-by-term convergent. One may check, however, that by carefully redistributing some of the terms, one obtains convergent sums that can be approximated by the corresponding continuous integrals. Proceeding in this manner, and using a numerical integrator, we obtained the results plotted in figure 3.

We observe that the form factor series converge in the following sense: while the one-particle terms agree with the exact result for ϵ greater than about $3k_BT$, the approximation including two-particle terms reaches the exact curve at ϵ around $2k_BT$. For energies $\epsilon \ll k_BT$, the thermal factors do not efficiently suppress many-particle contributions, and the convergence of the form factor expansion is expected to be slow.

We remark that the asymptotic behaviour for $\epsilon \gg k_BT$ of the two-particle terms is

$$h^{(2,0)}(\epsilon) \sim c_2 e^{-\beta\epsilon} \qquad h^{(1,1)}(\epsilon) \sim c_{1/2} e^{-\beta\epsilon} \qquad (5.20)$$

with

$$c_2 = -2 \int_0^\infty d\epsilon_1 \, \bar{n}_2(\epsilon_1) \qquad c_{1/2} = \int_0^\infty d\tilde{\epsilon}_1 \, \bar{n}_{1/2}(\tilde{\epsilon}_1) \qquad (5.21)$$

remarkably, the duality relation (2.8) leads to the relation

$$c_2 = -c_{1/2} \tag{5.22}$$

meaning that the Boltzmann tails of the two-particle terms cancel precisely. This 'conspiracy' was needed as, numerically, it is seen that the deviation between the exact curve $h(\epsilon)$ and the one-particle term $h^{(1,0)}(\epsilon)$ is far smaller than the individual Boltzmann tails of $h^{(2,0)}$ and $h^{(1,1)}$.

6. Conclusions

Summarizing the results collected in this paper, we have made some first steps on the way to realizing a computational scheme where the T dependence of physical observables in a fqH system (charge transport properties, in particular) is computed with direct reference to fractional statistics of the fundamental quasi-particles. We expect that on the basis of the formalism presented here, meaningful claims about the observability of the fractional statistics of CFT edge quasi-particles can be formulated. We leave this most interesting aspect for further study.

   We remark the continuum (CFT) limit of the CS model provides an ideal testing ground for form factor expansions for finite-temperature correlation functions, such as discussed in section 5.3 and in the literature [11, 12]. This is because, on the one hand, the theory is explicitly regularized by the finite extent of the spatial direction and, on the other hand, the finite-temperature Green functions are known from standard CFT considerations.

Appendix A. Algebraic properties of $\nu = \frac{1}{2}$ edge operators

The charged edge operators $J = J^-$, $J^\dagger = J^+$ in the edge theory at $\nu = \frac{1}{2}$ are part of an $SU(2)_1$ affine symmetry algebra. Together with the charge density $Q = i\sqrt{p}\partial\varphi$ they satisfy the commutation relations

$$
\begin{aligned}
[J^+_{m_2}, J^-_{m_1}] &= m_1\delta_{m_2+m_1} + Q_{m_2+m_1} \\
[Q_{m_2}, J^\pm_{m_1}] &= \pm 2J^\pm_{m_2+m_1} \qquad [Q_{m_2}, Q_{m_1}] = 2m_1\delta_{m_2+m_1}.
\end{aligned}
\tag{A.1}
$$

The fractionally charged edge quasi-particles $\phi^\pm$ transform in the spin-$\frac{1}{2}$ representation of the $SU(2)$ symmetry

$$[J^\pm_m, \phi^\pm_s] = 0 \qquad [J^\pm_m, \phi^\mp_s] = \pm\phi^\pm_{m+s} \qquad [Q_m, \phi^\pm_s] = \pm\phi^\pm_{m+s}. \tag{A.2}$$

Among themselves, the modes of $\phi^\pm$ satisfy so-called generalized commutation relations, which have been studied in the context of the spinon formulation of the $SU(2)_1$ CFT [28, 29].

Appendix B. Jack polynomials and Jack operators

In this appendix we briefly introduce the Jack polynomials that are used in sections 3 and 4 of this paper. We essentially follow the conventions of Iso [9], but we introduce a different notation for the coordinate-dependent Jack polynomials $P^{\beta}_{\{\mu\}}(\{z_i\})$ and the bosonic mode Jack operators $J^{\beta}_{\{\mu\}}\left(\left\{\frac{a_{-n}}{\sqrt{\beta}}\right\}\right)$.

We start by specifying an inner product on the ring of symmetric polynomials,

$$\langle p_{\{\lambda\}} | p_{\{\mu\}} \rangle_{\beta} = \delta_{\{\lambda\},\{\mu\}} \beta^{-l(\{\lambda\})} z_{\lambda} \tag{B.1}$$

where $p_{\{\lambda\}}(z_i) = \prod_{j=1}^{l(\{\lambda\})} p_{\lambda_j}(\{x_i\})$ with $p_{\lambda_j}(\{x_i\}) = \sum_i x_i^{\lambda_j}$ is the power sum set by a Young tableau $\{\lambda\} = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, $z_{\{\lambda\}}$ is $\prod_{i \geqslant 1} i^{l_i} l_i!$ with l_j the number of entries in $\{\lambda\}$ which satisfy $\lambda_i = j$ and β a rational number.

The coordinate Jack polynomials $P^{\beta}_{\{\lambda\}}(\{z_i\})$ are symmetric functions in the coordinates $\{z_i\}$ labelled by a Young tableau $\{\lambda\}$ and a rational number β. They are defined by the following properties.

Orthogonality:

$$\langle P^{\beta}_{\{\lambda\}}(\{z_i\}) | P^{\beta}_{\{\nu\}}(\{z_i\}) \rangle_{\beta} = \delta_{\{\lambda\},\{\nu\}} j^{\beta}_{\{\nu\}}.$$

Triangularity:

$$P^{\beta}_{\{\lambda\}}(\{z_i\}) = \sum_{\{\mu\}} v_{\lambda,\mu}(\beta) m_{\{\mu\}}$$

where $v_{\lambda,\mu}(\beta) = 0$ unless $\{\mu\} \leqslant \{\lambda\}$.

Normalization: the coefficient $v_{\lambda,\lambda} = 1$.

In this definition, $m_{\{\lambda\}}(\{z_i\})$ are the monomial symmetric functions $\sum_{\sigma} \prod_i z_i^{\lambda_{\sigma(i)}}$, where $\sum_{\sigma}$ denotes the sum over all permutations of the indices i. The partial ordering $\leqslant$ on partitions is the so-called dominance ordering on partitions of equal weight ($|\lambda| = |\mu|$): $\{\lambda\} \leqslant \{\mu\} \Leftrightarrow \sum_i = 1^j \lambda_i \leqslant \sum_{i=1}^{j} \mu_j$ for all j. The function $j^{\beta}_{\{\nu\}}$ in the inner product can be shown to be given by [23]

$$j^{\beta}_{\{\nu\}} = \prod_{(i,j)\in\{\nu\}} \frac{\beta(\nu'_j - i) + \nu_i - j + 1}{\beta(\nu'_j - i + 1) + \nu_i - j}. \tag{B.2}$$

In a notation where Jack polynomials are written as functions of power sums p_n, they satisfy a duality between $\beta = p$ and $\beta = \frac{1}{p}$

$$P^{p}_{\{\lambda'\}}\left(\left\{\frac{p_n}{p}\right\}\right) = (-1)^{|\lambda|} j^{p}_{\{\lambda\}} P^{1/p}_{\{\lambda\}}(-\{p_n\}) \tag{B.3}$$

where $\{\lambda'\}$ is the Young tableau dual to $\{\lambda\}$. It follows that

$$j^{p}_{\{\nu\}} j^{1/p}_{\{\nu'\}} = 1. \tag{B.4}$$

The following elementary property of the Jack polynomials:

$$\prod_{i,j}(1 - x_i y_j) = \sum_{\{\lambda\}} (-1)^{|\lambda|} P^{\beta}_{\{\lambda\}}(\{x_i\}) P^{1/\beta}_{\{\lambda'\}}(\{y_j\}) \tag{B.5}$$

can be used to rewrite expressions involving vertex operators.

For integer β an alternative inner product [23] on the Jack polynomials $P_{\{\mu\}}^{\beta}$ depending on only a finite set of coordinates $\{z_i\} = \{z_1, \ldots, z_n\}$ is given by

$$\langle\langle P_{\{\nu\}}^{\beta}(\{z_i\}) \,|\, P_{\{\mu\}}^{\beta}(\{z_i\})\rangle\rangle = \left(\prod_{i=1}^{n} \oint \frac{\mathrm{d}z_i}{2\pi\mathrm{i}} \frac{1}{z_i}\right) \Delta^{\beta}(\{z_i^{-1}\})\Delta^{\beta}(\{z_i\}) P_{\{\nu\}}^{\beta}(\{z_i^{-1}\}) \, P_{\{\mu\}}^{\beta}(\{z_i\})$$

(B.6)

where $\Delta^{\beta}(\{x_i\}) = \prod_{i<j}(x_i - x_j)^{\beta}$ denotes a generalized Vandermonde determinant. Although it is also possible to define this inner product for fractional β [23], we will use it in this form for integer β. The Jack polynomials are orthogonal with respect to this alternative inner product.

For the product of N quasi-hole vertex operators $\phi(z_i)$, the following expression can be derived:

$$\phi(z_1)\ldots\phi(z_N)|q\rangle = \Delta^{1/p}(\{z_i\}) \sum_{\{\lambda\}}(-1)^{|\lambda|} P_{\{\lambda'\}}^{1/p}(\{z_j\}) \, J_{\{\lambda\}}^{p}\left(\left\{\frac{a_{-n}}{\sqrt{p}}\right\}\right) \prod_{j=1}^{N} z_j^{q/p}|q+N\rangle \quad (B.7)$$

where we wrote $J_{\{\lambda\}}^{p}\left(\left\{\frac{a_{-n}}{\sqrt{p}}\right\}\right)$ for a Jack polynomial in which power sums p_n are replaced by bosonic modes, writing $a_n = (\partial\varphi)_n$. (We refer to such expressions as Jack operators.) Similarly,

$$J(z_1)\ldots J(z_N)|q\rangle = \Delta^{p}(\{z_i\}) \sum_{\{\lambda\}}(-1)^{|\lambda|} P_{\{\lambda'\}}^{p}(\{z_j\}) \, J_{\{\lambda\}}^{1/p}(\{\sqrt{p}a_{-n}\}) \prod_{j=1}^{N} z_j^{-q}|q-Np\rangle. \quad (B.8)$$

For brevity, we sometimes drop the explicit reference to the bosonic modes and write

$$J_{\{\lambda\}}^{1/p} \equiv J_{\{\lambda\}}^{1/p}(\{\sqrt{p}a_{-n}\}) \qquad J_{\{\lambda\}}^{p} \equiv J_{\{\lambda\}}^{p}\left(\left\{\frac{1}{\sqrt{p}}a_{-n}\right\}\right). \quad (B.9)$$

Appendix C. Proof of selection rules

We present a proof of the form factor selection rules of section 4.3.

(a) This is a consequence of energy conservation and can be found from the product of three delta functions,

$$\delta_{|n|+m,\lambda_1}\delta_{\lambda_1+|\rho|,|\mu|}\delta_{|n|+|\rho|,|\nu|} \quad (C.1)$$

present implicitly in equation (4.29).

(b) Proposition 2.4 in [22] states that it is possible to rewrite any Jack polynomial as a linear combination of products of Jack polynomials labelled with horizontal strips $\mathcal{J}_{\{\lambda\}}^{p} = \prod_i P_{(\lambda_i)}^{p}$,

$$P_{\{\lambda\}}^{p} = \sum_{\{\sigma\}\geqslant\{\lambda\}} \tilde{q}_{\{\lambda\}\{\sigma\}}\mathcal{J}_{\{\sigma\}}^{p}. \quad (C.2)$$

An important difference between this expansion and the expansion of a Jack polynomial in monomial symmetric functions is that the sum runs over tableaux $\{\sigma\}$ satisfying $\{\sigma\} \geqslant \{\lambda\}$ instead of $\{\sigma\} \leqslant \{\lambda\}$. From repeated application of proposition 5.3 in [22] (see also section 4.3) to a product of two Jack polynomials, where one is expanded using the expansion above, it follows that the tableau labelling the non-expanded Jack polynomial is contained in every tableau labelling a Jack polynomial appearing in the product. Exchanging the roles we also see that the tableau labelling the other Jack polynomial is contained in these tableaux.

1. Combining this knowledge with the fact that by triangularity the monomial symmetric functions $m_{\{n\}}$ can be expanded in Jack polynomials,

$$m_{\{n\}} = \sum_{\{\tau\} \leqslant \{n\}} \tilde{v}_{\{n\}\{\tau\}} P^p_{\{\tau\}} \qquad (C.3)$$

and applying this to the coordinate inner product in equation (4.29) we find that $\{\rho\}$ is contained in $\{v\}$. The operator inner product shows that $\{\mu\}$ differs at most a horizontal strip from $\{\rho\}$ and thus $\{\rho\}$ contains $\{\tilde{\mu}\} = (\mu_2, \ldots, \mu_M)$. We can conclude that $\{v\}$ contains $\{\tilde{\mu}\}$ and we have obtained (b1).

2. We can extract extra information from examining this construction once more, under the addition of a horizontal strip the length of a column can grow with one box. If we now multiply the Jack polynomials $P^p_{\{\tau\}}$ appearing in the expansion of the monomial symmetric function $m_{\{n\}}$ with the $\mathcal{J}^p_{\{\sigma\}}$ appearing in the expansion of the Jack polynomial $P^p_{\{\rho\}}$ we find that the maximal difference in column length between $\{v\}$ and $\{\tau\}$ is $l(\{\rho\})$ from which we can conclude that the only columns in $\{v\}$ which have a length exceeding $l(\{\mu\}) \geqslant l(\{\rho\})$ are those columns for which the corresponding tableau labelling the monomial symmetric function has a column of non-zero length. Because only monomial symmetric functions with at most p non-zero columns appear we obtain (b2).

(c) This is a simple consequence of (a) and (b1).

(d) By definition the Jack polynomials can be expanded in monomial symmetric functions, so we have

$$P^p_{\{\rho\}} = \sum_{\{\sigma\} \leqslant \{\rho\}} v_{\{\rho\}\{\sigma\}} m_{\{\sigma\}}. \qquad (C.4)$$

If we now use this expansion $P^p_{\{\rho\}}$ and then multiply the resulting $m_{\{\sigma\}}$ in the expansion by $m_{\{n\}}$, then the products in the expansion will be linear combinations of $m_{\{\sigma'\}}$ satisfying $\{\sigma'\} \leqslant \{\sigma\} + \{n\}$,

$$m_{\{n\}} P^p_{\{\rho\}} = \sum_{\{\sigma\} \leqslant \{\rho\}} v_{\{\rho\}\{\sigma\}} m_{\{n\}} m_{\{\sigma\}}$$

$$= \sum_{\{\sigma'\} \leqslant \{\rho\} + \{n\}} u_{\{n\}\{\rho\}\{\sigma'\}} m_{\{\sigma'\}}. \qquad (C.5)$$

Expanding the $m_{\{\sigma'\}}$ as we did in the proof of (b) the sum over products of monomial symmetric functions can be rewritten in terms of Jack polynomials again,

$$m_{\{n\}} P^p_{\{\rho\}} = \sum_{\{\sigma\} \leqslant \{\rho\} + \{n\}} w_{\{n\}\{\rho\}\{\sigma\}} P^p_{\{\sigma\}} \qquad (C.6)$$

and we find that $\{v\}$ is smaller in the sense of dominance order than a tableau $\{\sigma\}$ of the form $\{\rho\} + \{n\}$. Since $n_i \leqslant p$ and $\rho_i \leqslant \mu_i$, the result (d) follows.

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